Averaged model and integrable limits in nonlinear double-periodic Hamiltonian systems

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We derive average propagation model for a nonlinear wave dynamics in media with periodically varying parameters considering a general case with different periods of the nonlinearity and dispersion oscillations. Applying quasi-identical canonical transformation we find the conditions when the averaged Hamiltonian dynamics is close to an integrable model. We apply general theory to the practical problem of optical signal transmission in fiber lines with short-scale dispersion management.

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I. INTRODUCTION

Propagation of powerful, high-frequency nonlinear wave in media with varying dispersion is a fundamental physical problem with a wide range of practical applications like, for instance, optical pulse transmission in dispersion-managed (DM) fiber lines [1], a stretched pulse generation in mode-locking fiber laser systems [2], propagation of high intensities beams in second order nonlinear media with periodic poling (see, e.g., Ref. [3] and references therein), evolution of soliton in a periodically modulated nonlinear wave guide and other applications. Of particular interest is an optical pulse transmission in fiber, that is a superb demonstration of practical application of the fundamental soliton theory [4–8]. The traditional path-averaged soliton propagation in fiber lines with uniform (or even weakly varying) dispersion is governed [7,9] by the integrable [4] nonlinear Schrödinger equation (NLSE). Integrability of the NLSE makes possible to apply powerful mathematical method of the inverse scattering transform [4] to a variety of practical problems (see, e.g., Refs. [5–8]). Experimental [9] (and even first commercial [10]) implementations of the multichannel soliton transmission have stimulated further studies in soliton theory. In particular, soliton propagation in the systems with large variations of dispersion and nonlinearity has recently been put in the focus of intensive research [1,2,9,12–21]. Traditional soliton solution of the NLSE with uniform dispersion and without loss realizes continuous balance between nonlinearity and dispersion. Variations of dispersion and nonlinearity make impossible in general case to support such balance continuously. Nevertheless, a balance between nonlinear effects and dispersion can be achieved *in average* over the compensation period. Because of the fundamental and practical importance of this problem, it is of interest to develop generic theoretical methods to analyze properties of the basic model that can be used in a variety of specific applications.

In this paper, generalizing our previous results we derive path-average model in a case of different (rational commensurable) periods of nonlinearity and dispersion oscillations. This general model includes all previously studied physical systems and also describes a new regime with short-scale dispersion oscillations. Applying quasi-identical transformation [11–13] we demonstrate that the averaged dynamics of high-frequency nonlinear wave in systems with large-amplitude, periodic variations of dispersion and nonlinearity can be described in some particular limits by the integrable NLSE. As a specific physical and practical application of the general analysis, we present path-averaged theory of DM transmission lines with different periods of the power variation and dispersion compensation.

II. BASIC MODEL AND QUASI-IDENTICAL TRANSFORM

Evolution (in z) of a high-frequency wave in medium with periodically varying dispersion and nonlinearity is governed by the NLSE with periodic coefficients that can be written in the Hamiltonian form

$$i\frac{\partial A}{\partial z}=\{A,H\}=\frac{\delta H}{\delta A^*}=-d(z)A_{tt}-\epsilon\, c(z)|A|^2A, \qquad (1)$$

with the Hamiltonian

$$H=\int\left\{d(z)|A_t|^2-\frac{\epsilon\, c(z)}{2}|A|^4\right\}dt, \qquad (2)$$

and the Poisson brackets are defined as

$$\{F,G\}=\int\left(\frac{\delta F}{\delta A}\frac{\delta G}{\delta A^*}-\frac{\delta F}{\delta A^*}\frac{\delta G}{\delta A}\right)dt. \qquad (3)$$

Here periodic function $d(Z)=\tilde{d}+\langle d\rangle$ ($\langle\tilde{d}\rangle=0$) describes varying dispersion with the period L (in real world units) and $c(Z)$ corresponds to power oscillations with the period Z_a. We consider a general case when L and Z_a are rational commensurable, namely, $nZ_a=mL=Z_0$ with integer n and m. This includes as particular limits all known and studied cases and allows us to describe a novel regime with short-scale ($L\ll Z_a$) management. The distance $z=Z/Z_0$ is normalized by a minimal common period Z_0 of the functions d and c and the averaging throughout the paper is over this period. In the normalized units 1-periodic d and c have basic periods $1/m$ and $1/n$, respectively. Small parameter $\epsilon=Z_0/Z_{NL}$ (see, e.g., Refs. [1] and [12]) is proportional to the pulse power (Z_{NL}

here is a characteristic nonlinear scale). True breathing soliton presents a solution of Eq. (1) of the form $A(z,t) = \exp(ikz)F(z,t)$ with a periodic function $F(z+1,t)=F(z,t)$. Of interest is to find a systematic way to describe a family of periodic solutions F with different quasi-momentum k. The basic idea suggested in Ref. [12] (see also [1,13,14,19] is to use a small parameter ϵ to derive path-averaged model that gives regular, leading order in ϵ, description of the breathing soliton. Averaging cannot be performed directly in Eq. (1) in the case of the large variations of $\tilde{d} \gg \langle d \rangle$. However, path-averaged propagation equation can be obtained in the frequency domain [12]. In this paper we show that in some important limits an averaged equation for periodic breathing pulse can be transformed to the *integrable* NLS equation.

First, to eliminate the periodic dependence of the linear part we apply following [12] the so-called Floquet-Lyapunov transformation [11]

$$A_\omega = \phi_\omega \exp\{-i\omega^2 R(z)\}, \quad \frac{dR(z)}{dz} = d(z) - \langle d \rangle. \tag{4}$$

Here, $A_\omega = A(\omega,z)$ is a Fourier transform of $A(t,z) = \int A_\omega \exp[-i\omega t]d\omega$. Important observation (will be used below) is that for a fixed amplitude of d amplitude of the variation of the function R decreases with the increase of m. It can be easily found that $\max[R(z)] \propto 1/m$. In the new variables the equation takes the form

$$i\frac{\partial \phi_\omega}{\partial z} - \langle d \rangle \omega^2 \phi_\omega + \epsilon \int G_{\omega 123}(z)\,\delta(\omega+\omega_1-\omega_2-\omega_3) \times \phi_1^* \phi_2 \phi_3 d\omega_1 d\omega_2 d\omega_3 = 0, \tag{5}$$

here $G_{\omega 123}(z) = c(z)\exp\{i\Delta\Omega R(z)\}$ is 1-periodic and $\Delta\Omega = \omega^2 + \omega_1^2 - \omega_2^2 - \omega_3^2$. Note that $G_{\omega 123}$ depends only on the specific combination of the frequencies given by the resonance surface $\Delta\Omega$. Both the Fourier and the Floquet-Lyapunov transform (4) are canonical and the transformed Hamiltonian H reads

$$H = \langle d \rangle \int \omega^2 |\phi_\omega|^2 d\omega - \varepsilon \int \frac{G_{\omega 123}}{2} \times \delta(\omega+\omega_1-\omega_2-\omega_3)\,\phi_\omega^* \phi_1^* \phi_2 \phi_3 d\omega d\omega_1 d\omega_2 d\omega_3 . \tag{6}$$

Now we apply Hamiltonian averaging. Let us make the following change of the variables

$$\phi_\omega = \exp[\{\varepsilon K, \ldots\}]\varphi_\omega = \varphi_\omega - \epsilon\{K,\varphi_\omega\} + \cdots = \varphi_\omega + \epsilon \int V_{\omega 123}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \varphi_1^* \varphi_2 \varphi_3 d\omega_1 d\omega_2 d\omega_3 + \cdots,$$

here the functional $K = 0.5\int V_{\omega 123}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \varphi_\omega^* \varphi_1^* \varphi_2 \varphi_3 d\omega d\omega_1 d\omega_2 d\omega_3$ and $V_{\omega 123}(z) = i\int_0^z [G_{\omega 123}(\tau) - T_{\omega 123}]d\tau + i\,V_{\omega 123}(0)$, $(\langle V_{\omega 123} \rangle = 0)$ with

$$T_{\omega 123} = \langle G_{\omega 123} \rangle = \int_0^1 G_{\omega 123}(z)dz = \int_0^1 c(z)\exp\{i\Delta\Omega R(z)\}dz. \tag{7}$$

In the leading order in ϵ, a path-averaged equation has the form

$$i\frac{\partial \varphi_\omega}{\partial z} - \langle d \rangle \omega^2 \varphi_\omega + \epsilon \int T_{\omega 123}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \varphi_1^* \varphi_2 \varphi_3 d\omega_1 d\omega_2 d\omega_3 = 0, \tag{8}$$

The corresponding averaged Hamiltonian H is

$$\langle H \rangle = \langle d \rangle \int \omega^2 |\varphi_\omega|^2 d\omega - \varepsilon \int \frac{T_{\omega 123}}{2} \times \delta(\omega+\omega_1-\omega_2-\omega_3)\,\varphi_\omega^* \varphi_1^* \varphi_2 \varphi_3 d\omega d\omega_1 d\omega_2 d\omega_3 . \tag{9}$$

The Hamiltonian averaging introduced here presents a regular way to calculate next-order corrections to the averaged model. From the Hamiltonian structure of the starting equation it is clear that the matrix element $T_{\omega 123}$ has the following symmetries $T_{\omega 123} = T_{1\omega 23} = T_{\omega 132} = T^*_{23\omega 1}$. Note that the Eq. (8) possesses the remarkable property. The matrix element $T_{\omega 123} = T(\Delta\Omega)$ is a function of $\Delta\Omega$ and on the resonant surface $\omega+\omega_1-\omega_2-\omega_3=0$, $\Delta\Omega = \omega^2+\omega_1^2-\omega_2^2-\omega_3^2=0$, both $T_{\omega 123}$ and its derivative over $\Delta\Omega$ are regular. This observation allows us to make the following quasi-identical-like transformation [13], which eliminates the variable part of the matrix element $T_{\omega 123}$

$$\varphi_\omega = a_\omega - \frac{\epsilon}{\langle d \rangle}\int \frac{T_0 - T_{\omega 123}}{\Delta\Omega} a_1^* a_2 a_3 \times \delta(\omega+\omega_1-\omega_2-\omega_3) d\omega_1 d\omega_2 d\omega_3, \tag{10}$$

where $T_0 = T(0)$. This transformation has no singularities. If the integral part in this transform is small compared with a_ω, then in the leading order we get for a_ω

$$i\frac{\partial a_\omega}{\partial z} - \langle d \rangle \omega^2 a_\omega + \epsilon \int T_0 \times \delta(\omega+\omega_1-\omega_2-\omega_3) a_1^* a_2 a_3 d\omega_1 d\omega_2 d\omega_3 = 0. \tag{11}$$

This is nothing more, but the integrable nonlinear Schrödinger equation written in the frequency domain. Obviously, this transformation is quasi-identical only if the integral in Eq. (10) is small compared with a_ω. This is not true in a general case and that is why, typical solution of Eq.(8) has a form [14] different from cosh-shaped NLSE soliton. However, if the kernel function in Eq. (10) is small

$$|S(\Delta\Omega)| = \left| \frac{T_0 - T_{\omega 123}(\Delta\Omega)}{\Delta\Omega} \right| \ll 1, \tag{12}$$

then the averaged model can be reduced to the NLSE. In other terms, this is a condition on the functions $c(z)$ and

$d(z)$ that makes possible quasi-identical transformation. If the condition (12) is satisfied we can express solutions of the Eq. (8), and, consequently, of the original Eq. (1) via solutions of the NLSE in the explicit form:

$$A(t,z)=\int a_\omega e^{\{-i\omega t-i\omega^2 R\}}d\omega+\epsilon\int W_{\omega 123}a_1^* a_2 a_3 \times\delta(\omega+\omega_1-\omega_2-\omega_3)d\omega_1 d\omega_2 d\omega_3 d\omega,$$

here $W_{\omega 123}(z)=[V_{\omega 123}-(T_0-T_{\omega 123})/\langle d\rangle\Delta\Omega]\exp\{-i\omega t-i\omega^2\times R(z)\}$ and a_ω is a solution of the NLSE (11).

III. PATH-AVERAGE MODEL FOR DOUBLE-PERIODIC DM SYSTEMS WITH $L>Z_a$ (LONG-HAUL FIBER TRANSMISSION LINES)

Now we apply developed above general theory to a particular, but important physical problem, namely we consider soliton transmission in fiber lines. In the optical applications the periodic functions $d(z)$ and $c(z)$ are

$$d(z)=\frac{\lambda_0^2 D(z)Z_0}{4\pi c_l t_0^2},$$

$$c(z)=P_0 Z_0\sigma\exp[-2Z_0\gamma(z-z_k)],$$

$$z_k\leqslant z<z_{k+1.} \qquad (13)$$

Here, t_0 is a characteristic time parameter, P_0 is a characteristic power, c_l is the speed of light, λ_0 is the operating wavelength, $D(z)$ is the dispersion coefficient varying along the fiber line; the nonlinear coefficient $\sigma=2\pi n_2/(\lambda_0 A_{eff})$ where n_2 is the nonlinear refractive index, A_{eff} is the effective fiber area; $\gamma=0.05\ln 10\alpha$ describes the fiber loss (with α in dB/km). The effects of the N point optical amplifiers deployed at $z_k=kZ_a/Z_0(k=0,1,\ldots,N-1)$ on the pulse power is accounted for through point transformations at z_k resulting in the periodic self-phase modulation (nonlinear) coefficient $c(z)$.

First, we consider the case $L\geqslant Z_a$. To be specific, let us consider as an example, two-step dispersion map with the amplification distance $Z_a\ km$ and dispersion compensation period $L=2M\times Z_a\ km=Z_0\ km$ $(m=1,n=2M)$. Dispersion $d(z)=d+\langle d\rangle$ if $0<Z<M\times Z_a=L/2$ and $d(z)=-d+\langle d\rangle$ if $M\times Z_a<Z<2M\times Z_a=L$. Mean-free function R defined above can be found as $R(z)=dz-d/4$ if $0<z<1/2$ and $R(z)=-d[z-1/2]+d/4$ if $1/2<z<1$. After some calculations, it can be found that the kernel of the function $T(\Delta\Omega)$ in such a system is

$$T(X)=\frac{G-1}{G\ln G}\frac{\sin[XM]}{M}\frac{1}{(1+[2X/\ln G]^2)}\left\{\frac{\cos[X]}{\sin[X]}+\frac{2X}{\ln G}\frac{G+1}{G-1}\right\},\quad X=\frac{\Delta\Omega Z_a d}{2L}=\frac{\Delta\Omega\, d}{4M}. \qquad (14)$$

Here, gain $G=\exp[2\gamma Z_a]$ (γ is a fiber loss). It is interesting to look at some particular limits in this general formula. First, if $d=0$ (uniform dispersion along the system) we reproduce the result of Mollenauer *et al.*: $T(\Delta\Omega)=(G-1)/(G\ln,G)$ and because T is a constant, path-averaged model is just the integrable NLS equation. Second limit is the so-called ''lossless'' model [15] $(\gamma=0)$. In this case, $T(\Delta\Omega)=\sin[\Delta\Omega\, d/4]/[\Delta\Omega\, d/4]$. We justify now the use of the ''lossless'' system [15] for modeling of the practical (with fiber loss) fiber transmission system. It is interesting that developed here theory confirms that the periodic amplification and dispersion compensation can be handled as separate problems, provided that amplification distance is substantially different from the period of dispersion map. This is illustrated by Fig. 1 where transfer function $T(\Delta\Omega)$ is shown by solid line for different M (here $Z_a=50$ km and $\alpha=0.21$ dB/km). It is seen that for $M>1$ function $T(\Delta\Omega)$ is indeed getting close and close to that one (shown by dotted line) for the ''lossless'' model $T(\Delta\Omega)=\sin[\Delta\Omega\, d/4]/[\Delta\Omega\, d/4]$ multiplied by the path-averaged factor $(G-1)/(G\ln G)$. For $M=3$ two curves are almost the same up to $X\approx 2$ and for $M=10$ the curves are about identical up to $X\approx 3$. Small deviation near the region $X\approx\pi$ corresponds to quasi-resonance due to the first term in the brackets and is not important for pulses with the spectral width of the order of one in the dimensionless units. Obtained result proves that the power budget and the dispersion mapping, effectively, can be handled separately in long-haul transoceanic optical communication systems where amplification distance is typi-

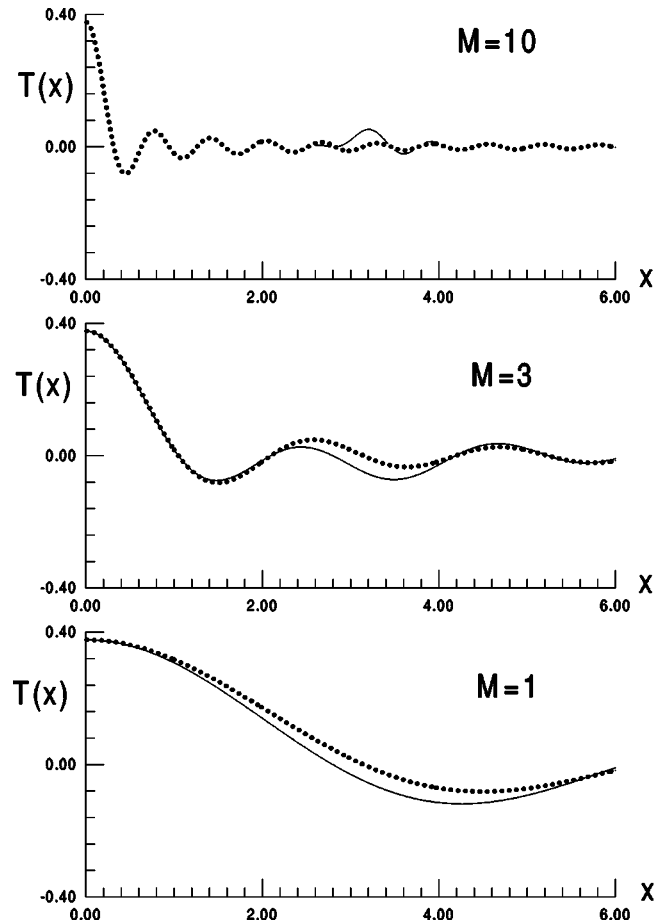

FIG. 1. Matrix element $T(X)$ (in the system with $L=2\,M\,Z_a\geqslant Z_a$) is plotted for different M: bottom $M=1$, middle $M=3$ and top $M=10$. Here amplification distance $Z_a=50$ km, $\alpha=0.21$ dB/km.

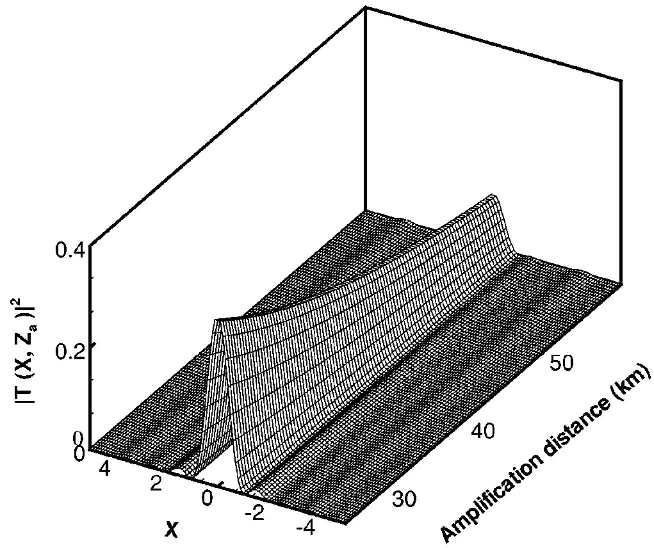

FIG. 2. Function $|T(X,Z_a)|^2$ for different X and Z_a in the interval from $Z_a=25$ km to 60 km. Maximum at $X=0$ slowly decays with Z_a, but for any amplification distance there are some points at which $T=0$.

cally much shorter than the dispersion compensation period. In Fig. 2 it, is shown function $|T(X,Z_a)|^2$ for different X and Z_a (the same loss as in Fig. 1) varying in the interval from 25 km to 60 km. Maximum at $X=0$ slowly decays with Z_a, but for any amplification distance there are some points at which $T=0$. Note that the points at which $T(\Delta\Omega)=0$ correspond to advanced dispersion maps with suppressed four-wave mixing [16].

IV. PATH-AVERAGE MODEL FOR DM SYSTEMS WITH SHORT-SCALE DISPERSION MANAGEMENT: $L \leqslant Z_a$

Next we consider a new regime with a short-scale ($L \ll Z_a$, and a general case $L \leqslant Z_a$) dispersion management. Optical fibers with $L \ll Z_a$ have recently been manufactured by Corning [17]. Traditional dispersion management typically assumes $L \geqslant Z_a$ (see, e.g., Refs. [15,12,18,19,14,20] and references therein). Recall that ultrashort, power-enhanced DM solitons in the traditional systems with $L \geqslant Z_a$ typically have too high power to be realized in practice [20]. It is of interest to find stable propagation regimes with short- and low-power solitons. As it has been shown in Ref. [21] rather short ($\leqslant 5$ ps) DM solitons in systems with the short-scale dispersion management could have low enough energy to provide for stable ultra high-bit-rate ($\geqslant 40$ Gb/s per channel) transmission. Here, we present a theory of solitons in systems with a short-scale management ($L \ll Z_a$) and demonstrate that a path average propagation in this regime, (even with the *large variations* of the dispersion) can be described by the integrable NLS equation. Again to be specific, let us consider a two-step dispersion map with the amplification distance $Z_a=Z_0$ $(n=1)$ and dispersion compensation period $L=Z_a/m$ km (or $1/m$ in the normalized units). Normalized dispersion $d(z)=d+\langle d\rangle$ if $k/m<z<(k+0.5)/m$ and $d(z)=-d+\langle d\rangle$ if $(k+0.5)/m<z<(k+1)/m$, here $k=0,1,2,\ldots,m-1$. Mean-free function R defined above can be found as $R(z)=d(z-k/m)-d/(4m)$ if $k/m<z<(k+0.5)/m$ and $R(z)=-d[z-k/m-1/(2m)]+d/(4m)$ if $(k+0.5)/m<z<(k+1)/m$. After some calculations, it can be shown that the matrix element $T_{\omega 123}$ in such a system is

$$T_{\omega 123}=\frac{G-1}{G\ln G}\cos[\Psi]\left\{\frac{\cos[Y-\Psi]\,G^{1/(2m)}-\cos[Y+\Psi]}{G^{1/(2m)}-1}\right.$$
$$\left.-i\,\frac{\sin[Y-\Psi]G^{1/(2m)}+\sin[Y+\Psi]}{G^{1/(2m)}+1}\right\}. \quad (15)$$

Here, $\Psi=\arctan[d\Delta\Omega/\ln G]=\arctan[4mY/\ln G]$, $Y=d\Delta\Omega/(4m)$. Function $T_{\omega 123}=T(Y)$ is plotted in Fig. 3 for different m. Here, $\alpha=0.21$ dB/km, $Z_a=60$ km. Real (top) and imaginary (bottom) parts of T are plotted for different m: $m=2$ (dashed line), $m=10$ (solid line) and $m=20$ (dotted line). For the real part of T one cannot see difference between three curves on this scale. Power of a parasitic signal occuring due to four-wave mixing is proportional to $|T|^2$ (Ref. [16]). In Fig. 4 it is plotted function $|T(Y)|^2$ versus Y. Here $m=10$ and $Z_a=40$ km. The formula (15) can be rewritten in the following form

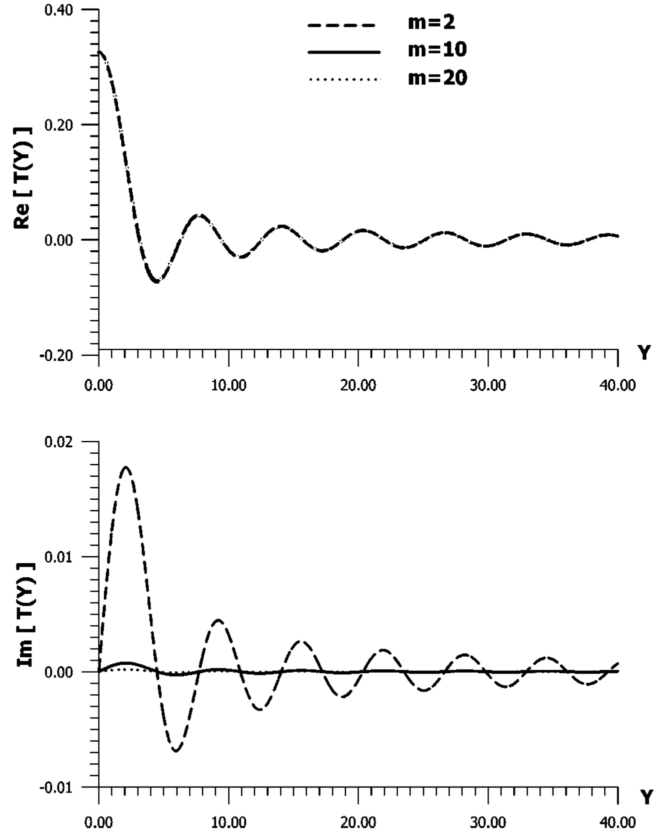

FIG. 3. Matrix element $T(Y)$ in the system with $L=Z_a/m<Z_a$. Real (top) and imaginary (bottom) parts of T are plotted for different m: dashed line $m=2$, solid line $m=10$ and dotted line $m=20$. On the top figure one cannot see difference between three curves on this scale. Here, $Z_a=60$ km, $\alpha=0.21$ dB/km.

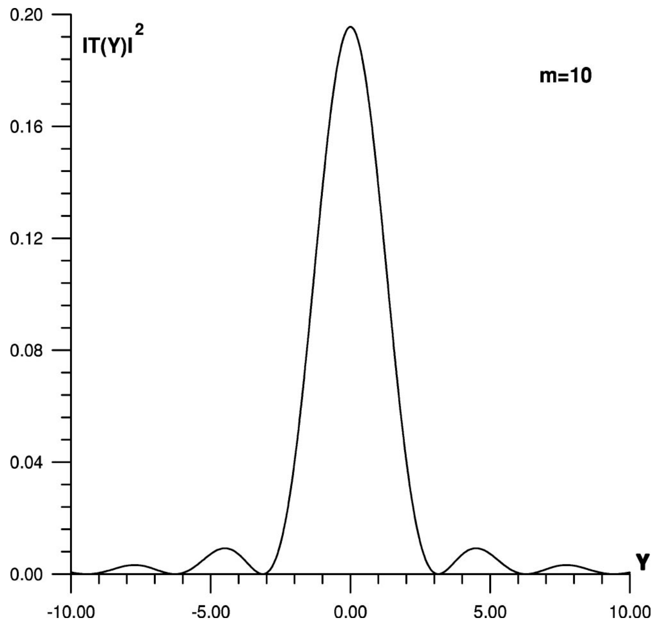

FIG. 4. Function $|T(Y)|^2$ versus Y for the system with $L=Z_a/m<Z_a=40$ km. Here $m=10$, $\alpha=0.21$ dB/km.

$$T(Y)=\frac{G-1}{G\ln G}\frac{1}{1+\left(\frac{4mY}{\ln G}\right)^2}\left\{\exp(-iY)+\frac{4mY}{\ln G}\left[\sin Y\frac{G^{1/(2m)}+1}{G^{1/(2m)}-1}+i\cos Y\frac{G^{1/(2m)}-1}{G^{1/(2m)}+1}\right]\right\}. \tag{16}$$

Next, we estimate the matrix element of the quasi-identical transformation

$$|S(\Delta\Omega)|\leqslant\left|\int_0^1\frac{c(z)[\exp(i\Delta\Omega R(z))-1]}{\Delta\Omega}dz\right|\leqslant\int_0^1|c(z)R(z)|dz\leqslant\max(R)\langle c\rangle=\frac{\langle c\rangle d}{4m}.$$

One can see that with increase of m (for the fixed other parameters) the path-averaged model (8) governing DM soliton propagation converges to the integrable NLS equation. It is seen also from Fig. 3 that with the increase of m the imaginary part of T decreases, but the real does not change significantly. It is interesting to note that in the limit of a very short-scale management (large m) we again get for T the lossless model approximation multiplied by the factor $(G-1)/(G\ln G)$: $T(Y)=\sin[\Delta\Omega d/(4m)]/[\Delta\Omega d/(4m)]*(G-1)/(G\ln G)$. However, increase of m (decrease of L) under the fixed characteristic bandwidth of the signal makes insignificant oscillatory structure of the kernel. This means that if $T(Y)$ is practically concentrated in some region ΔY, then for large m corresponding region in $d\,\Delta\Omega$ will be larger than for small m. For the pulses with the same spectral width this will mean that T is much flatter for large m and, as a matter of fact, for large m (small L) function T can be better approximated by a value $T(0)$. As a result, NLSE model works rather well in this limit and solution (of the path-averaged model) should be close to cosh-like soliton of the NLSE. Of course, the function $|S(Y)|$ increases with the growth of L and is not small in the opposite limit $Z_a \ll L$ (lossless model). Therefore, the shape of DM soliton in the lossless model [15] (and large variations of the effective dispersion) is not cosh as it is in the considered model. In contrast to the lossless model, evolution of soliton parameters over one period is highly asymmetric here due to loss. Rapid variations of the pulse width, peak power and chirp are accompanied by the exponential decay of the power due to loss. Nevertheless, numerical simulations have revealed that there exists a true periodic solution that reproduces itself at the end of the compensation cell (in this case - at the end of the amplification period). Note that though it is known that for the lossless model in the so-called weak map limit [15,12,18,19] DM soliton has shape close to cosh, this is not so obvious for system with loss and different periods of amplification and dispersion variations. It is intersting to mention that chirp-free points in this highly assymetrical map are pretty close to their positions in the lossless model - in the middle of fiber pieces: $Z_{CFP}=L/4$ or $3L/4$.

V. RESONANCE CASE $L=Z_a$

Now we consider in more detail an important limit $Z_a=L$. Consider for simplicity (more complicated systems can be analyzed in a similar manner) again ''symmetrical'' (recall that due to loss pulse evolution now is not symmetrical here as opposite to the lossless model) dispersion map with normalized dispersion $d(z)=d+\langle d\rangle$ if $0<z<1/2$ and $d(z)=-d+\langle d\rangle$ if $1/2<z<1$. Recall that distance is normalized here by L. Mean-free function R is $R(z)=d\,z-d/4$ if $0<z<1/2$ and $R(z)=-d[z-1/2]+d/4$ if $1/2<z<1$. The matrix element $T_{\omega 123}$ in this case reads

$$T_{\omega 123}=\frac{\exp[-\gamma L+id\Delta\Omega/4]-\exp[-id\Delta\Omega/4]}{-2\gamma L+id\Delta\Omega}+\frac{\exp[-2\gamma L-id\Delta\Omega/4]-\exp[-\gamma L+id\Delta\Omega/4]}{-2\gamma L-id\Delta\Omega}. \tag{17}$$

Here, $G=\exp[2\gamma L]$, $Z_0=Z_a=L$. After simple manipulations we get for matrix element $T_{\omega 123}$

$$T_{\omega 123}=\frac{G-1}{G\ln G}\frac{1}{1+\left(\frac{4Y}{\ln G}\right)^2}\left\{-\frac{8iY}{\ln G}\frac{\sqrt{G}}{G-1}\exp(iY)+\left[\frac{i4Y}{\ln G}\frac{G+1}{G-1}+1\right]\exp(-iY)\right\}. \tag{18}$$

Here, $\Psi=\arctan[d\Delta\Omega/\ln G]=\arctan[4Y/\ln G]$, $Y=d\Delta\Omega/4$. In another presentation, $T_{\omega 123}$ takes the form

$$T_{\omega 123}=\frac{G-1}{G\ln G}\cos[\Psi]\left\{\frac{\cos[Y-\Psi]\sqrt{G}-\cos[Y+\Psi]}{\sqrt{G}-1}-i\frac{\sin[Y-\Psi]\sqrt{G}+\sin[Y+\Psi]}{\sqrt{G}+1}\right\}. \tag{19}$$

Let us also mention for completeness that the formula (19) can also be rewritten as:

$$T(Y)=\frac{G-1}{G\ln G}\frac{1}{1+\left(\frac{4Y}{\ln G}\right)^2}\left\{\exp(-iY)+\frac{4Y}{\ln G}\left[\sin Y\frac{\sqrt{G}+1}{\sqrt{G}-1}+i\cos Y\frac{\sqrt{G}-1}{\sqrt{G}+1}\right]\right\}. \quad (20)$$

It is seen that $T(0)=(G-1)/(G\ln G)$ and $Y=0$ is the point of maximum of $|T(Y)|^2$. Similar, but a little bit more complicated formulas can be derived when transmission and compensating fibers have different losses and dispersions. The results for specific fiber systems (for instance, standard monomode fiber plus dispersion compensating fiber) will be published elsewhere.

VI. CONCLUSIONS

In conclusion, we have presented a general theory of DM soliton propagation in systems with different (rational commensurable) periods of the amplification and dispersion compensation. Using Hamiltonian averaging and quasi-identical transformation, we have shown that in some specific limits nonlinear wave propagation in system with periodically varying dispersion and nonlinearity can be described by the integrable NLS equation. As a particular physical application of the general theory, it is shown that the path-averaged propagation model in fiber systems with the short-scale dispersion management (when compensation period L is much shorter than the amplification distance Z_a) in the leading order is close to the integrable NLS equation. Derived formulas for matrix element $T_{\omega 123}$ play crucial role in the description of four-wave mixing [16]. This important application of the developed theory will be presented elsewhere.

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